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A method of solution of some elliptic P.D.E.'s

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SUMMARY
An extension of Lanczos' method employing "quadrature by differentiation" for the approximate solution of boundary value problems in ordinary differential equations is further extended to partial differential equations. The approximate solution is obtained in terms of piecewise polynomials or as rational function approximations. In the process, the boundary values are employed to yield initial values. In the illustrative problem treated, the method is combined with the matrix transformation method to yield, in the limit, exact initial values. Thus, we have a convenient method of invariant imbedding. The initial values thus obtained are utilized in the formulation of an eigenfunction solution to a non-separable problem in which the derivatives of the solution function are of interest, so that retention of analytic control is desirable.

1. Introduction

In [1], Lanczos introduces a method for the approximate solution of boundary value problems defined by ordinary differential equations. As pointed out by Lanczos, this method, based on „quadrature by differentiation", consists essentially of a power series expansion and truncation, in which the solution is expanded about the two boundary points. The point of truncation depends on the quadrature formula chosen. With regard to these quadrature formulas, it is important to note that the accuracy of the formula which contains $(n-1)$th derivatives of the integrand corresponds to that obtained by replacing of the integrand by its truncated power series of degree $(2n-1)$ (see [1]). The resultant polynomial approximation emphasizes accuracy at the boundary points, in contrast with Rayleigh–Ritz–Galerkin type procedures which are designed to optimize accuracy over the entire region of interest. Clearly, the Lanczos method, as presented in [1], would be unsuitable for problems in which the solution oscillates within the region of interest, as noted in [1].

A natural extension of Lanczos' method was treated in [2], where the region of interest was partitioned into a sufficient number of intervals so that the solution does not oscillate within any individual interval. The resultant approximate solution is obtained in terms of piecewise polynomials or as rational function approximations. In the present work, the method is extended to partial differential equations. The resultant solution is in the form of piecewise polynomials or rational functions in one of the independent variables, the coefficients being functions of the remaining independent variables. In the process of solution we make use of the matrix transformation method [3]. This illustrates the convenience of employing the matrix transformation method and the related method of summary representation [4], (which are discussed in [3] and [4], respectively, in connection with finite difference procedures) in connection with piecewise polynomial, or finite element, solutions. In addition, in [3] the matrix transformation method is confined to symmetric matrices. Here, the method is applied to a problem with an asymmetric matrix.

In the process of obtaining the solution, by the method illustrated here, we obtain initial data for the problem, so that the original boundary value problem may now be treated as an initial value problem. Moreover, as the number of intervals in the partition is increased, the accuracy of the initial data increases, so that an appropriate limit-taking procedure leads to the exact solution for the initial data. Thus we have a simple, formal method in contrast with some ad hoc methods of invariant imbedding (see, for example, [5]). Generally, the goal of invariant imbedding is to obtain a convenient numerical procedure for computing the solution function. But in certain problems, e.g. plate theory [5], torsion and plane elasticity problems [6], quantities of interest, i.e. the stresses, are given in terms of the derivatives of the solution. Hence, it is desirable to retain analytic control of the solution. We, therefore, illustrate the use of the initial data to obtain an eigenfunction series solution. For illustrative purposes, simple two-dimensional problems are used.

2. Deflection of rectangular membrane

Consider a rectangular membrane of length a and width b, subjected to uniform tension S (Fig. 1). This problem is chosen because it is simple, so that essential procedures are not obscured by difficulties inherent in the problem, and the solution is known, providing a check on the results obtained here. The differential equation for the deflection w is:

$$w'' + \ddot{w} = Q. \tag{1}$$

Here, the dash $(\ )'$ denotes differentiation with respect to x, $x = \bar{x}/a$; the dot $(\dot{\ })$. denotes differentiation with respect to y, $y = \bar{y}/a$, $\bar{x}$ and $\bar{y}$ being the dimensional coordinates; $Q = -a^2 p/S$, p being the transverse pressure. The boundary conditions are:

$$w(0, y) = w(1, y) = w(x, 0) = w(x, \rho) = 0 \tag{2}$$

where $\rho = b/a$.

We will now illustrate the extension of Lanczos' method, as extended in [2], to partial differential equations. We do not employ here the modification introduced in [2] because our ultimate goal here is to obtain exact initial values by a limit-taking process, and, for this purpose, the original method is satisfactory. Partition the rectangle into N strips, so that

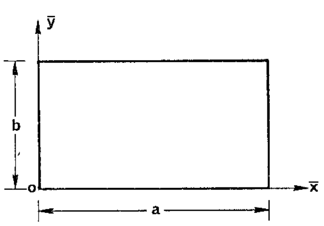

Figure 1. Rectangular membrane.

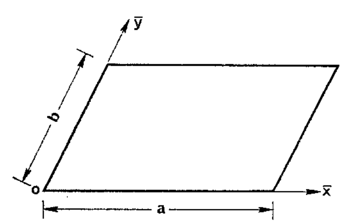

Figure 2. Parallelogram-shaped membrane.

$$x_n = \frac{n}{N};\ x_0 = 0;\ x_N = 1;\ x_{n+1} - x_n = \frac{1}{N}. \tag{3}$$

Applying the approximate quadrature formula [1] to a typical strip and retaining derivatives to w''', we have

$$\begin{aligned} w_{n+1} - w_n &= \tfrac{1}{2}(w'_{n+1} + w'_n)N^{-1} - \tfrac{1}{10}(w''_{n+1} - w''_n)N^{-2} + \tfrac{1}{120}(w'''_{n+1} + w'''_n)N^{-3}, \\ w'_{n+1} - w'_n &= \tfrac{1}{2}(w''_{n+1} + w''_n)N^{-1} - \tfrac{1}{12}(w'''_{n+1} - w'''_n)N^{-2}. \end{aligned} \tag{4}$$

Here (to illustrate the notation), $w'_n = w'(x, y)|_{x=x_n}$. From (1),

$$w'' = Q - \ddot{w},\ w''' = Q' - \ddot{w}'. \tag{5}$$

Let k be an integer and let

$$Q = Q_k \sin k\pi x,\ Q_{kn} = Q_k \sin k\pi x_n. \tag{6}$$

Introduce the differentiation operator D and the shift operator E,

$$D = \frac{d}{dy};\ Ew_n = w_{n+1};\ E^{-1}w_n = w_{n-1}. \tag{7}$$

Then, from (5), (6) and (7), (4) may be written as

$$\begin{aligned} &(E-1)\left(1 - \frac{N^{-2}}{10}D^2\right)w_n - \frac{N^{-1}}{2}(E+1)\left(1 - \frac{N^{-2}}{60}D^2\right)w'_n \\ &\quad = -\frac{N^{-2}}{10}(E-1)Q_{kn} + \frac{N^{-3}}{120}(E+1)Q'_{kn}, \\ &\frac{N^{-1}}{2}(E+1)D^2 w_n + (E-1)\left(1 - \frac{N^{-2}}{12}D^2\right)w'_n = \frac{N^{-1}}{2}(E+1)Q_{kn} - \frac{N^{-2}}{12}(E-1)Q'_{kn}, \\ &\qquad n = 0, 1, 2, \ldots, N-1. \end{aligned} \tag{8}$$

The preceding $2N$ equations plus the conditions $w_0 = w_N = 0$ determine the $2(N+1)$ unknowns. The boundary conditions are [noting the last two conditions in (2)]

$$w_n(0) = w_n(\rho) = w'_n(0) = w'_n(\rho) = 0. \tag{9}$$

Eliminating w'_n between the two equations in (8), we obtain

$$[L_1(E^2+1) - 2L_2E]w_n = N^{-2}[\tfrac{1}{4}(E+1)^2 - \tfrac{1}{10}(E-1)^2]Q_{kn} - \frac{N^{-3}}{30}(E^2-1)Q'_{kn},$$
$$n = 0, 1, 2, \ldots, N-2, \tag{10}$$

where

$$L_1 = 1 + \frac{N^{-2}}{15}D^2 + \frac{N^{-4}}{240}D^4,\ L_2 = 1 - \frac{13N^{-2}}{120}D^2 + \frac{N^{-4}}{80}D^4. \tag{11}$$

In addition to the first two boundary conditions in (9), we obtain from (5) and (6) and the last two conditions in (2)

$$\ddot{w}_n(0) = \ddot{w}_n(\rho) = Q_{kn}. \tag{12}$$

The $(N-1)$ equations in (1) plus $w_0 = w_N = 0$ and the boundary conditions in (9) and (12) determine the w_n.

For $k = 1$ and $N = 2$, we have

$$D^4 w_1 - \tfrac{416}{3} D^2 w_1 + 1280 w_1 = -Q_1\left(112 + \frac{16\pi}{3}\right) \tag{13}$$

with boundary conditions

$$w_1(0) = w_1(\rho) = 0, \;\; \ddot{w}_1(0) = \ddot{w}_1(\rho) = Q_1. \tag{14}$$

The solution is

$$\begin{aligned} w_1 = {} & 0.1006 Q_1 \left[\cosh 3.153y + \frac{1-\cosh 3.153\rho}{\sinh 3.153\rho} \sinh 3.153y - 1\right] \\ & - 2.115 \times 10^{-6} Q_1 \left[\cosh 11.35y + \frac{1-\cosh 11.35\rho}{\sinh 11.35\rho} \sinh 11.35y - 1\right], \end{aligned} \tag{15}$$

and from the second equation in (8), taking into account that $w_1' = 0$, we have

$$\begin{aligned} w_0' = {} & 0.3154 Q_1 \left[\cosh 3.153y + \frac{1-\cosh 3.153\rho}{\sinh 3.153\rho} \sinh 3.153y - 1\right] \\ & + 40.47 \times 10^{-6} Q_1 \left[\cosh 11.35y + \frac{1-\cosh 11.35}{\sinh 11.35} \sinh 11.35y\right]. \end{aligned} \tag{16}$$

For comparison, the exact solutions are:

$$\begin{aligned} w_1 &= \frac{Q_1}{\pi^2}\left[\cosh \pi y + \frac{1-\cosh \pi\rho}{\sinh \pi\rho} \sinh \pi y - 1\right], \\ w_0' &= \frac{Q_1}{\pi}\left[\cosh \pi y + \frac{1-\cosh \pi\rho}{\sinh \pi\rho} \sinh \pi y - 1\right]. \end{aligned} \tag{17}$$

Thus we see that satisfactory approximations for w_n and w_n' are obtained even when $N = 2$. Note that the solutions in (15) and (16) contain very small spurious components which can be shown to remain small for all y within the range of interest.

For $N > 2$, the form of the differential equations (10) and the boundary conditions (9) and (12) is such that, for any N, the matrix transformation method [3] can be employed to obtain w_n (this approach will be demonstrated later). Then (8) can be used to solve for w_n'. The higher order derivatives are then obtained with the aid of (5) and (8). Hence we have, for $x_n \leqq x \leqq x_{n+1}$, the polynomial approximations

$$\begin{aligned} w &= w_n + w_n'(x - x_n) + \tfrac{1}{2} w_n''(x - x_n)^2 + \tfrac{1}{6} w_n'''(x - x_n)^3, \\ w &= w_{n+1} + w_{n+1}'(x - x_{n+1}) + \tfrac{1}{2} w_{n+1}''(x - x_{n+1})^2 + \tfrac{1}{6} w_{n+1}(x - x_{n+1})^3. \end{aligned} \tag{18}$$

Clearly, the first equation in (18) will be more accurate near $x = x_n$, the second near $x = x_{n+1}$.

Following Lanczos [1], we can obtain improved accuracy by formulating the solution in terms of rational function approximations. Consider a point x_* in the vicinity of x_{n+1}.

For definiteness, let $x_n < x_* < x_{n+1}$, and let $w_* = w(x_*, y)$. The approximate quadrature equations over the range $[x_*, x_{n+1}]$ are:

$$\begin{aligned} w_{n+1} - w_* &= \tfrac{1}{2}(x_{n+1} - x_*)(w'_{n+1} + w'_*) - \tfrac{1}{10}(x_{n+1} - x_*)^2(w''_{n+1} - w''_*) \\ &\quad + \tfrac{1}{120}(x_{n+1} - x_*)^3(w'''_{n+1} + w'''_*), \\ w'_{n+1} - w'_* &= \tfrac{1}{2}(x_{n+1} - x_*)(w''_{n+1} + w''_*) - \tfrac{1}{12}(x_{n+1} - x_*)^2(w'''_{n+1} - w'''_*). \end{aligned} \tag{19}$$

Rearranging, taking into account (5), we obtain

$$\begin{aligned} & w_* - \tfrac{1}{10}(x_{n+1} - x_*)^2\ddot{w}_* + \tfrac{1}{2}(x_{n+1} - x_*)w'_* - \tfrac{1}{120}(x_{n+1} - x_*)^3\ddot{w}'_* \\ &\quad = w_{n+1} - \tfrac{1}{2}(x_{n+1} - x_*)w'_{n+1} + \tfrac{1}{10}(x_{n+1} - x_*)^2(Q_{n+1} - Q_* - \ddot{w}_{n+1}) \\ &\quad - \tfrac{1}{120}(x_{n+1} - x_*)^3(Q'_{n+1} + Q'_* - \ddot{w}'_{n+1}) - \tfrac{1}{2}(x_{n+1} - x_*)\ddot{w}_* + w'_* \\ &\quad - \tfrac{1}{12}(x_{n+1} - x_*)^2\ddot{w}'_*, \\ & w'_* - \tfrac{1}{2}(x_{n+1} - x_*)\ddot{w}_* - \tfrac{1}{12}(x_{n+1} - x_*)^2\ddot{w}'_* \\ &\quad = w'_{n+1} - \tfrac{1}{2}(x_{n+1} - x_*)(Q_{n+1} + Q_* - \ddot{w}_{n+1}) \\ &\quad + \tfrac{1}{12}(x_{n+1} - x_*)^2(Q'_{n+1} - Q'_* - \ddot{w}'_{n+1}). \end{aligned} \tag{20}$$

The preceding equations determine w_* and w'_*. We need only the particular solution. As an example, consider the case of $N = 2$, to which the solution is given by (15) and (16). Note that, from symmetry, $w'_1 = \ddot{w}'_1 = 0$ and $w'_2 = -w'_0$. The second (negligible) terms in (15) and (16) will be omitted. Consider a point x_* in the vicinity of the point $x = \frac{1}{2}$, say $\frac{1}{4} < x_* < \frac{3}{4}$, and let

$$\kappa_1 = 0.10036; \;\; \kappa_2 = 0.3154; \;\; \kappa_3 = 3.153. \tag{21}$$

Then the combination of (15), (16), (20) and (21) yields

$$\begin{aligned} w_* &= \frac{Q_1\kappa_1[1 - \frac{7}{30}\kappa_3^2(x_1 - x_*)^2 - \frac{1}{240}\kappa_3^4(x_1 - x_*)^4]}{1 + \frac{1}{15}\kappa_3^2(x_1 - x_*)^2 + \frac{1}{240}\kappa_3^4(x_1 - x_*)^4} \\ &\quad \cdot \left[\cosh\kappa_3 y + \frac{1 - \cosh\kappa_3\rho}{\sinh\kappa_3\rho}\sinh\kappa_3 y\right] \\ &\quad + Q_1\left[-\kappa_1 + \tfrac{1}{20}(x_1 - x_*)^2(7 + 3\sin\pi x_*) + \frac{3\pi}{40}(x_1 - x_*)^3\cos\pi x_*\right], \\ w'_* &= \frac{Q_1\kappa_1\kappa_3^2(x_1 - x_*)}{1 + \frac{1}{15}\kappa_3^2(x_1 - x_*)^2 + \frac{1}{240}\kappa_3^4(x_1 - x_*)^4} \\ &\quad \cdot \left[\cosh\kappa_3 y + \frac{1 - \cosh\kappa_3\rho}{\sinh\kappa_3\rho}\sinh\kappa_3 y\right] \\ &\quad - \tfrac{1}{2}Q_1\left[(x_1 - x_*)(1 + \sin\pi x_*) + \frac{\pi}{6}(x_1 - x_*)^2\cos\pi x_*\right]. \end{aligned} \tag{22}$$

Since x_* is an arbitrary point in the vicinity of x_1, the subscripted asterisk may be dropped. The piecewise polynomial and rational function approximate solutions obtained by this method satisfy the boundary conditions at $x = 0$ and $x = 1$, but the boundary conditions at $y = 0$ and $y = \rho$ are satisfied only at the discrete points $x = x_n$, $n = 0, 1, \ldots, N$.

Having obtained an approximation for w_0', we can now treat the problem as an initial value problem. Thus, the preceding serves as a simple formal approximation procedure for invariant imbedding [5]. As N becomes large, the number of points along $y = 0$ and $y = \rho$ at which the boundary conditions are satisfied increases. In the limit, we would expect to obtain the exact solution for w_0'. In preparation for such a limiting procedure, we return to (8), where the grouping of terms is such that their order of magnitude is clear, and we retain terms of order $1/N$ only. Note that the functions w_n, Q_{kn} and their derivatives as well as $(E+1)f_n$ are all of order unity, whereas $(E-1)f_n$ is of order $1/N$. Thus we obtain

$$\begin{aligned} &(E-1)w_n - \frac{N^{-1}}{2}(E+1)w_n' = 0, \\ &\frac{N^{-1}}{2}(E+1)D^2 w_n + (E-1)w_n' = \frac{N^{-1}}{2}(E+1)Q_{kn}, \qquad n = 0, 1, 2, \ldots, N-1. \end{aligned} \tag{23}$$

The preceding equations are also obtainable directly from a one-term quadrature approximation. Eliminating w_n' between the two equations in (23), we obtain (with a shift in the index n)

$$\begin{aligned} &(D^2 + 4N^2)(w_{n+1} + w_{n-1}) - 2(-D^2 + 4N^2)w_n \\ &\quad = 2Q_k\left(1 + \cos\frac{k\pi}{N}\right)\sin k\pi x_n, \qquad n = 1, 2, \ldots, N-1. \end{aligned} \tag{24}$$

The preceding equations plus $w_0 = w_N = 0$ and the first two boundary conditions in (9) determine the solution. The right side of (24) was obtained with the aid of (6) and trigonometric identities.

Following Zuber [3], the problem will now be solved by the matrix transformation method. In matrix form, (24) may be written as

$$L_3 T w - 2L_4 w = 2Q_k\left(1 + \cos\frac{k\pi}{N}\right)s, \tag{25}$$

where

$$L_3 = (D^2 + 4N^2),\ L_4 = (-D^2 + 4N^2), \tag{26}$$

T is the $(N-1) \times (N-1)$ matrix

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 \\ 1 & 0 & 1 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & 1 & . & . & . & . \\ 0 & . & . & . & . & 1 & 0 & 1 \\ 0 & . & . & . & . & 0 & 1 & 0 \end{bmatrix}, \tag{27}$$

and w and s are the vectors

$$w = [w_1, w_2, \ldots, w_{N-1}]^T, \quad s = \left[\sin\frac{k\pi}{N},\ \sin\frac{2k\pi}{N},\ \ldots,\ \sin\frac{(N-1)k\pi}{N}\right]^T. \tag{28}$$

The eigenvalues λ_m and the unit eigenvectors $\boldsymbol{p}_m$ of T are, respectively, [3]

$$\lambda_m = 2\cos\frac{m\pi}{N},\ \boldsymbol{p}_m = \left(\frac{2}{N}\right)^{\frac{1}{2}}\left[\sin\frac{m\pi}{N},\ \sin\frac{2m\pi}{N},\ \ldots,\ \sin\frac{(N-1)m\pi}{N}\right]^T,$$
$$m = 1, 2, \ldots, N-1. \quad (29)$$

Let Λ be the matrix of the eigenvalues and P the matrix of the eigenvectors. Note that $P = P^T$, $T = P\Lambda P$, and $PP = I$, where I is the unit matrix. Premultiplying (25) by P, we obtain

$$L_3\Lambda P\boldsymbol{w} - 2L_4P\boldsymbol{w} = 2Q_k\left(1+\cos\frac{k\pi}{N}\right)P\boldsymbol{s} \quad (30)$$

Let

$$P\boldsymbol{w} = \boldsymbol{z},\ P\boldsymbol{s} = \boldsymbol{r}. \quad (31)$$

Then we have $(N-1)$ uncoupled equations

$$L_3\lambda_m z_m - 2L_4 z_m = 2Q_k\left(1+\cos\frac{k\pi}{N}\right)r_m, \quad m = 1, 2, \ldots, N-1, \quad (32)$$

where

$$\begin{aligned} z_m &= \left(\frac{2}{N}\right)^{\frac{1}{2}}\sum_{j=1}^{N-1}\sin\frac{mj\pi}{N}w_j,\\ r_m &= \left(\frac{2}{N}\right)^{\frac{1}{2}}\sum_{j=1}^{N-1}\sin\frac{mj\pi}{N}\sin\frac{k\pi x_j}{N}. \end{aligned} \quad (33)$$

Transformation of the first two boundary conditions in (9) yields

$$z_m(0) = z_m(\rho) = 0. \quad (34)$$

The solution to (32), taking into account (26) and (34) is

$$z_m = A_m\cosh\alpha_m y + B\sinh\alpha_m y - \frac{Q_kN^{-2}}{2}\left(1+\cos\frac{k\pi}{2}\right)\frac{r_m}{2-\lambda_m} \quad (35)$$

where

$$\begin{aligned} \alpha_m &= 2N\left(\frac{2-\lambda_m}{2+\lambda_m}\right)^{\frac{1}{2}},\ A_m = \frac{Q_kN^{-2}}{2}\left(1+\cos\frac{k\pi}{N}\right)\left(\frac{r_m}{2-\lambda_m}\right),\\ B_m &= \frac{Q_kN^{-2}}{2}\left(1+\cos\frac{k\pi}{N}\right)\left(\frac{1-\cosh\alpha_m\rho}{\sinh\alpha_m\rho}\right)\left(\frac{r_m}{2-\lambda_m}\right). \end{aligned} \quad (36)$$

Inverting to the w_n, noting that

$$\boldsymbol{w} = P\boldsymbol{z}, \quad (37)$$

we have

$$w_n = Q_kN^{-3}\left(1+\cos\frac{k\pi}{N}\right)\sum_{j=1}^{N-1}\sum_{m=1}^{N-1}\frac{C_m-1}{2-\lambda_m}\sin\frac{nm\pi}{N}\sin\frac{mj\pi}{N}\sin k\pi x_j, \quad (38)$$

where

$$C_m = \cosh \alpha_m y + \frac{1 - \cosh \alpha_m \rho}{\sinh \alpha_m \rho} \sinh \alpha_m y. \tag{39}$$

Now, from (23), we have

$$w_1' + w_0' = 2N(w_1 - w_0), \; w_1' - w_0' = \frac{N^{-1}}{2} (Q_{k1} + Q_{k0} - \ddot{w}_1 - \ddot{w}_0). \tag{40}$$

Subtracting the second equation in (40) from the first, retaining only terms of order unity, and noting that $w_0 = 0$, we have

$$w_0' = N w_1. \tag{41}$$

To evaluate the summation on j in (38), note that $j/N = x_j$ and, therefore, as N becomes infinite

$$\sum_{j=1}^{N-1} \sin \frac{mj\pi}{N} \sin k\pi x_j \left(\frac{1}{N}\right) = \int_0^1 \sin m\pi x \sin k\pi x dx$$
$$= \begin{cases} 0 \text{ for } m \neq k, \\ \frac{1}{2} \text{ for } m = k. \end{cases} \tag{42}$$

So that we have, from (38), (41) and (42),

$$w_0' = \tfrac{1}{2} Q_k N^{-1} \left(1 + \cos \frac{k\pi}{N}\right)\left(\sin \frac{k\pi}{N}\right)\left(\frac{C_k - 1}{2 - \lambda_k}\right). \tag{43}$$

In addition, as N goes to infinity,

$$1 + \cos \frac{k\pi}{N} = 2; \; \sin \frac{k\pi}{N} = \frac{k\pi}{N}; \; 2 - \lambda_k = \frac{k^2 \pi^2}{N^2}; \; \alpha_k = k\pi. \tag{44}$$

The last two of the preceding equations follow from (29) and (36), respectively. From (39) and (44), (43) becomes

$$w_0' = \frac{Q_k}{k\pi}\left(\cosh k\pi y + \frac{1 - \cosh k\pi\rho}{\sinh k\pi\rho} \sinh k\pi y - 1\right) \tag{45}$$

which coincides with the exact solution.

3. Parallelogram-shaped membrane

Consider a parallelogram-shaped membrane (Fig. 2) with an edge of length a in the x-direction and an edge of length b in the y-direction. Here x and y are nondimensional skew coordinates with an included angle α; $x = \bar{x}/a$, $y = \bar{y}/a$, $\bar{x}$ and $\bar{y}$ being the dimensional coordinates. The membrane is under uniform tension S. The differential equation for the deflection w is:

$$w'' - 2\beta \dot{w}' + \ddot{w} = Q. \tag{46}$$

and the boundary conditions are:

$$w(0, y) = w(1, y) = w(x, 0) = w(x, \rho) = 0 \tag{47}$$

where

$$\beta = \cos\alpha; \; Q = (pa^2/S)\gamma^2; \quad \gamma = \sin\alpha. \tag{48}$$

The remaining symbols play the same roles here as in the preceding section.

The differential equation (46) does not have a complementary solution which is the product of arbitrary functions of x alone and y alone, but it does possess a product solution of the form

$$w_c = e^{\lambda x} W(y). \tag{49}$$

The reduced differential equation for $W(y)$ is

$$\ddot{W} - 2\beta\lambda\dot{W} + \lambda^2 W = 0. \tag{50}$$

The eigenfunctions satisfying (50) and the last two conditions in (47) are:

$$W_n = A_n e^{\lambda_n \beta y} \sin \lambda_n \gamma y, \tag{51}$$

where the A_n are arbitrary constants, and the eigenvalues are

$$\lambda_n = \pm \frac{n\pi}{\gamma\rho}, \quad n = 1, 2, 3, \ldots . \tag{52}$$

To derive an orthogonality relation for these eigenfunctions, note that the differential equation (50) for W_n may be written in vector form

$$\lambda_n \begin{bmatrix} 0 & 1 \\ 1 & -2\beta D \end{bmatrix} \begin{bmatrix} Y_n \\ W_n \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & D^2 \end{bmatrix} \begin{bmatrix} Y_n \\ W_n \end{bmatrix} = 0, \tag{53}$$

where

$$Y_n = \lambda_n W_n. \tag{54}$$

The orthogonality relation, obtained in the usual manner, is for $\lambda_m \neq \lambda_n$

$$\int_0^\rho [Y_m, W_m] \begin{bmatrix} 0 & 1 \\ 1 & -2\beta D \end{bmatrix} \begin{bmatrix} Y_n \\ W_n \end{bmatrix} dy = 0. \tag{55}$$

A particular solution of (46) satisfying the last two conditions in (47) is

$$W_p = \tfrac{1}{2} Q(y^2 - \rho y). \tag{56}$$

Hence, a solution

$$w = \sum_{n=-\infty}^{\infty} e^{\lambda_n x} W_n(y) + \tfrac{1}{2} Q(y^2 - \rho y) \tag{57}$$

satisfies the last two boundary conditions in (47). Suppose that w_0' were found by the method of invariant imbedding. Note that, from (57) and (54),

$$w_0' = \sum_{n=-\infty}^{\infty} Y_n(y). \tag{58}$$

Then the first two boundary conditions in (47) may be written as:

$$\sum_{n=-\infty}^{\infty} \begin{bmatrix} Y_n \\ W_n \end{bmatrix} + \begin{bmatrix} 0 \\ w_p \end{bmatrix} = \begin{bmatrix} w_0' \\ 0 \end{bmatrix}, \tag{59}$$

where w_p is given by (56). The A_n [see (51)] may then be found with the aid of the orthogonality relation (55).

We will, therefore, find w_0' in the same manner as in the preceding section. As before, we will seek the solution for a region partitioned into N strips, considering N to be large, and then take the limit as N goes to infinity. Hence, we begin by considering one-term quadrature approximations

$$\begin{aligned} w_{n+1} - w_n &= \tfrac{1}{2}(w_{n+1}' + w_n')N^{-1}, \\ w_{n+1}' - w_n' &= \tfrac{1}{2}(w_{n+1}'' + w_n'')N^{-1}, \qquad n = 0, 1, 2, \ldots, N-1. \end{aligned} \tag{60}$$

Taking into account (46) and eliminating w_n', we obtain

$$\begin{aligned} &(D^2 - 4\beta ND + 4N^2)w_{n+1} + 2(D^2 - 4N^2)w_n \\ &\quad + (D^2 + 4\beta ND + 4N^2)w_{n-1} = 4Q, \qquad n = 1, 2, \ldots, N-1. \end{aligned} \tag{61}$$

In addition, we have

$$w_0 = w_N = 0. \tag{62}$$

The $(N+1)$ equations (61) and (62) plus the boundary conditions (47) determine the w_n. Here, the operators for w_{n+1} and w_{n-1} are no longer identical, so that the method of [4] is no longer applicable to the differential equation as a whole. However, we may obtain a set of complementary solutions

$$w_{cn} = H_n e^{\mu y}. \tag{63}$$

Then the homogeneous form of (61) yields

$$\begin{aligned} &(\mu^2 - 4\beta N\mu + 4N^2)H_{n+1} + 2(\mu^2 - 4N^2)H_n \\ &\quad + (\mu^2 + 4\beta N\mu + 4N^2)H_{n-1} = 0, \qquad n = 1, 2, \ldots, N-1. \end{aligned} \tag{64}$$

In addition, we assume that (62) is satisfied separately by the complementary and particular solutions. Thus,

$$H_0 = H_n = 0. \tag{65}$$

Let

$$\mu^2 - 4\beta N\mu + 4N^2 = \xi, \ \mu^2 - 4N^2 = \zeta, \ \mu^2 + 4\beta N\mu + 4N^2 = \eta. \tag{66}$$

Assuming a solution to (64) of the form $H_n = \tau^n$, we obtain

$$H_n = K_1\tau_1^n + K_2\tau_2^n, \tag{67}$$

where K_1 and K_2 are arbitrary constants and

$$\tau_{1,2} = \frac{-\zeta \pm (\zeta^2 - \xi\eta)^{\frac{1}{2}}}{\xi}. \tag{68}$$

Enforcing the conditions (65) leads to

$$K_2 = -K_1, \ \tau_{1m} = \tau_{2m} e^{\pm 2m\pi i/N}, \quad m = 1, 2, \ldots, N-1. \tag{69}$$

The plus sign in the exponent of e in the immediately preceding equation is the only one which corresponds to

$$\zeta_m = -(\xi_m \eta_m)^{\frac{1}{2}} \cos\left(\frac{m\pi}{N}\right) \tag{70}$$

which satisfies (64) and yields

$$H_{nm} = K_m (\eta_m/\xi_m)^{n/2} \sin\left(\frac{nm\pi}{N}\right). \tag{71}$$

From (66) and (70), we can write

$$\left(\frac{\mu_m}{2N}\right)^2 - 1 = -\left\{\left[1 + \left(\frac{\mu_m}{2N}\right)^2\right]^2 - \beta^2 \left(\frac{\mu_m}{N}\right)^2\right\}^{\frac{1}{2}} \cos\left(\frac{m\pi}{N}\right). \tag{72}$$

In anticipation of the limiting process, note that, for $N \to \infty$, (72) yields

$$\mu_m = \pm \frac{m\pi}{\gamma}. \tag{73}$$

The preceding equation takes into account (48).

The $\boldsymbol{H}_m$, $m = 1, 2, \ldots, N-1$, where $\boldsymbol{H}_m = [H_{1m}, H_{2m}, \ldots, H_{(N-1)m}]^T$, are the eigenvectors (more precisely the right-hand eigenvectors) of the matrix of equations (64). If we denote the matrix of the equations in (64) by $[M]$, then its left-hand eigenvectors, which are biorthogonal to the $\boldsymbol{H}_m$, are the (right-hand) eigenvectors of $[M]^T$ [7]. From (64) and (67), it is clear that the two sets of eigenvectors have the same form but with ξ and η interchanged. If $\boldsymbol{u}_m$ and $\boldsymbol{v}_m$ are normalized right-hand and left-hand eigenvectors of $[M]$, respectively, then

$$\begin{aligned} \boldsymbol{u}_m &= R_m \left[(\xi_m/\eta_m)^{\frac{1}{2}} \sin\frac{m\pi}{N}, \ldots, (\zeta_m/\eta_m)^{(N-1)/2} \sin\frac{(N-1)m\pi}{N} \right]^T, \\ \boldsymbol{v}_m &= R_m \left[(\eta_m/\xi_m)^{\frac{1}{2}} \sin\frac{m\pi}{N}, \ldots, (\eta_m/\xi_m)^{(N-1)/2} \sin\frac{(N-1)m\pi}{N} \right]^T, \\ \boldsymbol{v}_m^T \boldsymbol{u}_m &= 0 \text{ for } m \neq n. \end{aligned} \tag{74}$$

To find R_m, note that for $m = n$

$$\boldsymbol{v}_m^T \boldsymbol{u}_m = R_m^2 \sum_{n=1}^{N-1} \sin^2 \frac{nm\pi}{N} = R_m^2 \left(\frac{N}{2}\right) = 1; \quad R_m = (2/N)^{\frac{1}{2}}. \tag{75}$$

The expression for the summation in (75) is derived in [4], p. 39.

The desired particular solutions w_{pn} to (61) are independent of y. The matrix of the resultant equations is symmetric and the solution may be obtained by employing the method of [3] as in the preceding section. The result is

$$w_{pn} = (2QN^{-3}) \sum_{m=1}^{N-1} \sum_{j=1}^{N-1} \sin \frac{nm\pi}{N} \sin \frac{mj\pi}{N} (\lambda_m - 2)^{-1} \tag{76}$$

where $\lambda_m = 2\cos(m\pi/N)$, as in the preceding section.

We can now write the solutions for w_n in matrix form. Let

$$\begin{aligned} &e_m = e_{1m} \cosh \mu_m y + e_{2m} \sinh \mu_m y, \ \boldsymbol{e} = \boldsymbol{e}(y) = [e_1, e_2, \ldots, e_{N-1}]^T, \\ &\boldsymbol{w}_p = [w_{p1}, w_{p2}, \ldots, w_{p(N-1)}]^T, \ \boldsymbol{w} = \boldsymbol{w}(y) = [w_1, w_2, \ldots, w_{N-1}]^T, \\ &U = [\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{N-1}], \ V = [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{N-1}]. \end{aligned} \tag{76}$$

Then

$$\boldsymbol{w}(y) = U\boldsymbol{e}(y) + \boldsymbol{w}_p. \tag{77}$$

From the last two conditions in (47),

$$\boldsymbol{w}(0) = \boldsymbol{w}(\rho) = 0. \tag{78}$$

Substituting (77) into (78) and premultiplying the result by V^T, we obtain e_{1m} immediately and can solve for e_{2m}. Thus,

$$e_{1m} = -\boldsymbol{v}_m^T \boldsymbol{w}_p = \sum_{i=1}^{N-1} v_{im} w_{pi}, \ e_{2m} = \frac{1 - \cosh \mu_m \rho}{\sinh \mu_m \rho} e_{1m}. \tag{79}$$

To obtain w_0', note that, from (60) and (62), to order N^{-1}

$$w_0' = N w_1, \tag{80}$$

as in the preceding section. As N goes to infinity, the solution (80) may be considerably simplified. We begin by simplifying w_{pn} in (76). Note that for $N \to \infty$

$$\begin{aligned} &N^{-1} \sum_{j=1}^{N-1} \sin \frac{mj\pi}{N} = \int_0^1 \sin m\pi x \, dx = \begin{cases} (2/m\pi), & m \text{ odd}, \\ 0, & n \text{ even}, \end{cases} \\ &N^2(\lambda_m - 2) = 2N^2 \left(\cos \frac{m\pi}{N} - 1 \right) = -m^2\pi^2, \ N \sin \frac{m\pi}{N} = m\pi. \end{aligned} \tag{81}$$

Hence

$$\begin{aligned} w_{pn} &= -4Q \sum_{m=1,3,5}^{N-1} (m\pi)^{-3} \sin \frac{nm\pi}{N}, \\ N w_{p1} &= -4Q \sum_{m=1,3,5}^{\infty} (m\pi)^{-2} = -\tfrac{1}{2}Q. \end{aligned} \tag{82}$$

Moreover, for $N \to \infty$, $\xi_m/\eta_m = 1$ [see (66)], so that from (74), (76), (77), (79) and (82),

$$w_{c1} = 8QN^{-1} \sum_{i=1}^{N-1} \sum_{k=1,3,\ldots}^{N-1} \sum_{m=1}^{N-1} (k\pi)^{-3} \sin\frac{m\pi}{N} \sin\frac{im\pi}{N} \sin\frac{ik\pi}{N} \times \left(\cosh \mu_m y + \frac{1-\cosh \mu_m \rho}{\sinh \mu_m \rho} \sinh \mu_m y\right). \tag{83}$$

Noting that

$$N^{-1} \sum_{i=1}^{\infty} \sin\frac{im\pi}{N} \sin\frac{ik\pi}{N} = \int_0^1 \sin m\pi x \sin k\pi x \, dx = \begin{cases} \frac{1}{2}, & \text{for } m = k, \\ 0, & \text{for } m \neq k, \end{cases} \tag{84}$$

then we have, from (81), (83) and (84),

$$Nw_{c1} = 4Q \sum_{m=1}^{\infty} (m\pi)^{-2} \left(\cosh \mu_m y + \frac{1-\cosh \mu_m \rho}{\sinh \mu_m \rho} \sinh \mu_m y\right), \tag{85}$$

where $\mu_m = m\pi/\gamma$, as in (73). Taking into account (82) and (85), (80) becomes

$$w_0' = Q\left[4 \sum_{m=1,3,\ldots}^{\infty} (m\pi)^{-2} \left(\cosh \mu_m y + \frac{1-\cosh \mu_m \rho}{\sinh \mu_m \rho} \sinh \mu_m y\right) - \tfrac{1}{2}\right]. \tag{86}$$

4. Discussion

If $\alpha = \pi/2$, i.e. $\gamma = 1$ so that $\mu_m = m\pi$, then (86) yields the rectangular membrane solution. Note that, unlike the solution in (51), which is restricted in form by the assumption of (49) and which must hold in the entire parallelogram-shaped region, (86) has to hold only in an infinitesimally wide strip near $x = 0$. Therefore, it is unaffected by the fact that $\alpha \neq \pi/2$. Indeed, (86) coincides in form with the solution for a rectangular membrane of length $a \sin \alpha$ and width b. Substitution of (56) and (86) into (59), taking into account (55) yields the coefficients A_n in the eigenfunction expansion [see (51)]. The resultant solution is exact in the usual series solution sense, i.e. by taking a sufficient number of terms we can make the error as small as we please. However, in a problem which is tractable by separation of variables, e.g. the rectangular membrane, a series may be constructed so that each term in the series satisfies all boundary conditions. Whereas for nonseparable problems, such as the parallelogram-shaped membrane, some boundary conditions (at $x = 0$ and $x = 1$ in our case) are only satisfied by the infinite series as a whole.

Usually, the method of invariant imbedding is employed to change a boundary value problem into an initial value problem, which is more convenient for obtaining a numerical solution. In some problems, quantities of interest are related to the derivatives of the solution. For example, if we set $(p/S) = 2$ [see (46) and (48)], then (46) and (47) define the Prandtl stress function for the torsion of a prismatic bar [6] in which the stresses are related to the derivatives of the solution function denoted here by w. It is therefore desirable to retain analytic control of the solution. This goal is attained by the eigenfunction solution which we obtained. A similar situation exists in some problems of plane elasticity (e.g. [8]) and plate theory [5].

In conclusion, a method introduced by Lanczos for the approximate solution of boundary value problems in ordinary differential equations has been extended to boundary value problems in partial differential equations. It has been employed in obtaining approximate solutions and as a method of invariant imbedding to obtain approximate and exact initial values. The initial values were then utilized to obtain an eigenfunction solution to a non-separable problem in which the derivatives of the solution function are of interest, so that retention of analytic control is desirable in order to avoid numerical differentiation.

REFERENCES

[1] C. Lanczos, *Applied Analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1956.

[2] H. Herman, Extension of Lanczos' method of fundamental eigenvalue approximation (to be published).

[3] R. Zuber, Matrix transformation method of approximate solution of partial differential equations, *Zastosowania Matematyki*, 11 (1970) 177–193.

[4] G. N. Polozhii, *The Method of Summary Representation for Numerical Solution of Problems of Mathematical Physics*, Pergamon Press, Oxford, 1965.

[5] E. Angel, N. Distefano and J. Anil, Invariant imbedding and the reduction of boundary value problems of thin plate theory to Cauchy formulations, *International Journal of Engineering Science*, 9 (1971) 933–945.

[6] I. S. Sokolnikoff, *Mathematical Theory of Elasticity*, 2nd ed., McGraw-Hill, New York, 1956.

[7] E. Isaacson and H. B. Keller, *Analysis of Numerical Methods*, Wiley, New York, 1966.

[8] P. J. Torvik, The elastic strip with prescribed end displacements, *J. Appl. Mech.*, 38 (1971) 929–936 (see "Introduction" for discussion of orthogonality problems; also, list of references).